

COUNTABILITY AND APPROACH THEORY

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ABSTRACT. In approach theory, we can provide arbitrary products of ∞p -metric spaces with a natural structure, whereas, classically only if we rely on a countable product and the question arises, then, whether properties which are derived from countability properties in metric spaces, such as sequential and countable compactness, can also do away with countability. The classical results which simplify the study of compactness in pseudometric spaces, which proves that all three of the main kinds of compactness are identical, suggest a further study of the category $p\mathbf{MET}^\infty$.

1. Introduction

In the paper we will be mainly working within the category $p\mathbf{MET}^\infty$ of extended pseudo-metric spaces and non expansive maps, which, as we know, is bicontractively embedded in the category \mathbf{AP} of approach spaces and contractions. We then first emphasize the fact that, in approach theory, we can provide arbitrary products of ∞p -metric spaces with a natural structure, viz. a notion of distance working no longer between pairs of points but rather between points and sets, whereas, classically, we only have that a product of metrizable spaces is metrizable only if we rely on a countable product. Secondly, if we can forget about the necessity to have only a countable product in approach theory and we can have a structure which is good for any kind of products, the question arises now whether properties which are derived from countability properties in metric spaces, such as sequential and countable compactness, can also do away with countability. Clearly, in approach theory, countability will still play a role as far as topological spaces are

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concerned because topological spaces are nicely embedded as a simultaneously concretely reflective and coreflective subconstruct of **AP**.

By depicting further relationships between categories in the frame of approach theory, we recall that the full subconstruct consisting of all the subspaces of products of ∞p metric approach spaces in **AP** (i.e., the epireflective hull of $p\mathbf{MET}^\infty$ in **AP**), is the category of uniform approach spaces, **UAP**.

2. Preliminaries

We shall use the following symbols :

$$\mathbb{R}_+ := [0, \infty[, \quad \mathbb{R}_+^* :=]0, \infty[, \quad \overline{\mathbb{R}_+} := [0, \infty].$$

$\mathbb{F}(X)$ will stand for the set of all filters on X , and $\mathbb{U}(X)$ will stand for the set of all ultrafilters on X . If F is a given filter on X , then we will denote by $\mathbb{F}(F)$ the collection of all filters on X which are finer than F , and by $\mathbb{U}(F)$ the collection of all ultrafilters on X which are finer than F .

We recall that, in an approach space X , the adherence operator is defined as

$$\alpha F(x) \doteq \sup_{F \in \mathbb{F}(X)} \delta(x, F), \quad \forall x \in X, \quad \forall F \in \mathbb{F}(X)$$

where $\delta : X \times 2^X \rightarrow \overline{\mathbb{R}_+}$ is the distance on X determining the approach structure.

Finally, if $|\mathbf{SET}|$ is the class of all sets and $X \in |\mathbf{SET}|$, we shall denote the set of all finite (resp. countable) subsets of X by $2^{(X)}$ (resp. $2^{((X))}$).

We recall also that a filter F on X is called *countable* if it has a filter base with a countable number of elements.

By $\mathbb{F}c(X)$ (resp. $\mathbb{F}_e(X)$) we denote the countable (resp. elementary) filters on X .

3. Countable and sequential compactness

In a way the concept of compactness arises from considerations of accumulation points of infinite sets. Applications of its contents compel a central technique of topology that of reducing and refining covers and various formulations : countable compactness, sequential compactness, Lindelöf property...

After having such concepts been put into the framework of approach spaces, a unified image was produced of compactness (countable compactness, sequential compactness) for topological spaces and of boundedness (total boundedness) for metric spaces.

Now, as far as these countability properties are concerned, we wonder in what sense countability is still an important property in approach theory and can differentiate between situations. We will be able to see that countability happens to be deeply inherent in the properties and results like those in this section can not be improved.

In [8] R. Lowen introduced the concept of a measure $m(X)$ (also noted here as $\mu(X)$) of compactness for an approach space X , as a canonical concept arising naturally from Kuratowski's measure of non-compactness for subsets of a complete metric space and, as one might expect, it was shown that $\mu(X) = 0$ for a compact topological approach space and $\mu(X) = \infty$ for a non-compact space.

Similarly, a measure of countable compactness $\overline{CC}(X)$ was introduced in [3] for which, in a topological approach space, a null value is translated in being countably compact. We recall,

“For $(X, (A(x))_{x \in X})$ an approach space, the following equivalent expressions give the measure of compactness:

- $\overline{CC}(X) = \sup_{(x_n)_{n \in \mathbb{N}} \in r(X)} \inf_{x \in X} \alpha(\{x_n\})(x)$
- $\overline{CC}_2(X) = \sup_{\phi \in \prod_{x \in X} A(x)} \sup_{(x_n)_{n \in \mathbb{N}} \in r(X)} \inf_{x \in X} \liminf_{n \rightarrow \infty} \phi(x)(x_n)$
- $\overline{CC}_3(X) = \sup_{F \in \mathbb{F}_e(X)} \inf_{x \in X} \alpha F(x)$
- $\overline{CC}_4(X) = \sup_{F \in \mathbb{F}_c(X)} \inf_{x \in X} \alpha F(x),$

where we put $r(X)$ for the set of all sequences on X .”

We have these formulas and they represent countable compactness but there is a characterization, well known from topological spaces, which is missing: how can we measure countable compactness in X by working with its sets of local distances or “neighbourhoods”?, i.e. by working with its approach system. We shall next provide it.

For an approach space X we define

$$\mu_{cc}(X) := \sup_{\psi \in \Psi} \inf_{K \in 2^{\Gamma_\psi}} \sup_{z \in X} \inf_{k \in K} \psi(k)(z)$$

where $\Psi = \{\psi : \Gamma_\psi \subset \mathbb{N} \rightarrow \cup_{x \in X} A(x) \mid \forall x, \exists n \in \Gamma_\psi : \psi(n) \in A(x)\}$.

THEOREM 3.1. For any approach space X :

$$\mu_{cc}(X) = \sup_{F \in \mathbb{F}_c(X)} \inf_{x \in X} \alpha F(x).$$

Proof. Put $m(X) := \sup_{F \in \mathbb{F}_c(X)} \inf_{x \in X} \alpha F(x)$. To show that $\mu_{cc}(X) \leq m(X)$ suppose that, for some $r \in \mathbb{R}_+$: $\mu_{cc}(X) > r$. Then there exists $\psi_0 \in \Psi$ such that for all $K \in 2^{(\Gamma\psi_0)}$: $\sup_{z \in X} \inf_{k \in \Gamma\psi_0} \psi_0(k)(z) > r$. Consequently, if for all $K \in 2^{(\Gamma\psi_0)}$, we put

$$F_K := \{z \in X \mid \inf_{k \in K} \psi_0(k)(z) > r\},$$

then it follows that $F_K \neq \emptyset$ and $F_{K_1} \cap F_{K_2} = F_{K_1 \cup K_2}$, for any $K, K_1, K_2 \in 2^{(\Gamma\psi_0)}$. Thus the collection $\{F_K \mid K \in 2^{(\Gamma\psi_0)}\}$ is a basis for a countable filter F^* . Since ψ_0 is an element in Ψ , we can make an arbitrary choice $x \mapsto n(x)$ such that $\psi_0(n(x)) \in A(x)$ and, hence, $\psi_0 \circ n \in \prod A(x)$. Then it follows that

$$\begin{aligned} m(X) &= \sup_{F \in \mathbb{F}_c(X)} \inf_{x \in X} \sup_{\varphi \in A(x)} \sup_{F \in F} \inf_{y \in F} \varphi(y) \\ &\geq \inf_{x \in X} \sup_{\varphi \in A(x)} \sup_{F \in F^*} \inf_{y \in F} \varphi(y) \\ &= \sup_{\phi \in \prod_X A(x)} \inf_{x \in X} \sup_{F \in F^*} \inf_{y \in F} \phi(x)(y) \\ &= \sup_{\phi \in \prod_X A(x)} \inf_{x \in X} \sup_{K \in 2^{(\Gamma\psi_0)}} \inf_{y \in F_K} \phi(x)(y) \\ &\geq \inf_{x \in X} \sup_{K \in 2^{(\Gamma\psi_0)}} \inf_{y \in F_K} \psi_0(n(x))(y) \\ &\geq \inf_{x \in X} \inf_{y \in F_{\{n(x)\}}} \psi_0(n(x))(y) \\ &\geq \inf_{x \in X} r = r. \end{aligned}$$

From the arbitrariness of r it follows that $\mu_{cc}(X) \leq m(X)$.

On the other hand, to show that $\mu_{cc}(X) \geq m(X)$, suppose that for some $r \in \mathbb{R}_+$: $m(X) > r$. Then there exists a countable filter $F_0 = \langle \{F_n \mid n \in \mathbb{N}\} \rangle$

and there exists $\phi_0 \in \prod_{x \in X} A(x)$ such that, for all $x \in X$,

$$\sup_{n \in \mathbb{N}} \inf_{y \in F_n} \phi_0(x)(y) > r,$$

which means that, for each $x \in X$, there exists $n_x \in \mathbb{N}$ with

$$F_{n_x} \subset \{z \in X \mid \phi_0(x)(z) > r\}.$$

First, put $\lambda : X \rightarrow \mathbb{N} : x \rightarrow n_x$ and $\Gamma := \{n \in \mathbb{N} \mid \lambda^{-1}(n) \neq \emptyset\}$.

As a consequence, for all $n \in \Gamma$:

$$F_n \subset \bigcap_{x \in \lambda^{-1}(n)} \{\phi_0(x) > r\}.$$

Secondly, put $\psi_0(n) := \inf_{x \in \lambda^{-1}(n)} \phi_0(x)$, for $n \in \Gamma$, and, clearly, since for all $x \in X : n_x \in \Gamma$, we know that $\psi_0 \in \Psi$.

Now, take $\mathbb{K} \in 2^{(\Gamma)}$ arbitrary and choose

$$z_0 \in \bigcap_{n \in \mathbb{K}} F_n \subset \bigcap_{n \in \mathbb{K}} \{z \in X \mid \psi_0(n)(z) \geq r\}.$$

Then $\inf_{n \in \mathbb{K}} \psi_0(n)(z_0) \geq r$ which means $\sup_{z \in X} \inf_{n \in \mathbb{K}} \psi_0(n)(z) \geq r$ and

$$\mu_{cc}(X) = \sup_{\psi \in \Psi} \inf_{\mathbb{K} \in 2^{(\Gamma_\psi)}} \sup_{z \in X} \inf_{k \in K} \psi(k)(z) \geq \inf_{\mathbb{K} \in 2^{(\Gamma)}} \sup_{z \in X} \inf_{n \in \mathbb{K}} \psi_0(n)(z) \geq r.$$

Again, from the arbitrariness of r it follows that $\mu_{cc}(X) \geq m(X)$. \square

It was already mentioned, but is easily deduced from the above result that, for topological spaces, $\mu_{cc}(X) = 0$ if and only if (X, τ) is countably compact.

We shall, next recall the definition of a third form of compactness [3] which sometimes it is presented for analysts, as the most important form of compactness in topological spaces.

DEFINITION 3.2. [3] For an approach space the measure of sequential compactness is defined as

$$\mu_{sc}(X) = \sup_{(x_n)_{n \in \mathbb{N}} \in r(X)} \inf_{k \uparrow : \mathbb{N} \rightarrow \mathbb{N}} \inf_{x \in X} \lambda < x_{k(n)} > (x).$$

Again we encounter a measure with properties which one might want it to have and, for a topological approach space X , we have $\mu_{sc}(X) \in \{0, \infty\}$ and, further, $\mu_{sc}(X) = 0$ if and only if every sequence in X has a converging subsequence.

Perfectly analogous to 6.1.4 and 6.1.5 from [10], it can be proved that, in the particular case of the category $p\mathbf{MET}^\infty$ we attain $\mu_{cc}(X) = \mu_{sc}(X) = 0$ if and only if (X, d) is totally bounded and, within the category $pq\mathbf{MET}^\infty$ of extended pseudo-quasi-metric spaces and non expansive maps, $\mu_{cc}(X) < \infty$, and $\mu_{sc}(X) < \infty$ if and only if (X, d) is bounded.

In general the only implications among the three kinds of compactness—namely μ_c , μ_{cc} and μ_{sc} —are that both μ_c and μ_{sc} are bigger than μ_{cc} .

Further, the relationships between compactness, countable compactness and sequential compactness in topological spaces can provide us with topological approach spaces which show that the inequalities mentioned are in general strict. The next proposition from [3] proves that there are other relationships in special cases, as we will also have the opportunity to see later in Section 4.

PROPOSITION 3.3. [3] *For a first countable approach space X , the measures of countable and sequential compactness coincide.*

3.1. Invariance properties

We devote this subsection to the investigation of the basic structural questions about continuous images and products for which the measure of compactness and countable and sequential compactness feature generalizations of known properties in **TOP**.

In [8] and [3] it is shown that the Tychonoff theorem can be generalized for approach spaces in the following way:

If $(X_j, A_j)_{j \in J} \subset |\mathbf{AP}|$ then,

$$\mu_c\left(\prod_{j \in J} X_j\right) = \sup_{j \in J} \mu_c(X_j).$$

and, for approach spaces X_i , $i \in \mathbb{N}$,

$$\mu_{sc}\left(\prod_{i \in \mathbb{N}} X_i\right) = \sup_{i \in \mathbb{N}} \mu_{sc}(X_i).$$

Further, in [3], it is mentioned that the Novak space provides an example of a countably compact topological space with the characteristic that the product of this space with itself is not a countably compact space.

As a conclusion we can only state that, for products of approach spaces, and since the projections are contractions, the measures of countable compactness of the components are always less than or equal to the corresponding measure for the product (see Theorem 3.15, [3]).

We do have a somewhat more restrictive result for this notion, that is to say that, for first countable approach spaces $(X_i)_{i \in \mathbb{N}}$,

$$\mu_{cc}\left(\prod_{i \in \mathbb{N}} X_i\right) = \sup_{i \in \mathbb{N}} \mu_{cc}(X_i).$$

Further, we would like to investigate whether countable compactness, sequential compactness and boundeness admit similar properties in **AP**

as those known in **TOP** under the formation of continuous images.

First, we recall a concept in **AP** similar to the concept of an open map in **TOP**.

DEFINITION 3.4. [11] Let X and Y be approach spaces. Let $f : X \rightarrow Y$ be a map and $x \in X$. Then f is expansive at x if and only if for all $\varphi \in A(x)$, $f(\varphi)$ belongs to $A'(f(x))$, (where $f(\varphi)(y) = \inf_{y=f(z)} \varphi(z)$).

As it was expected, for (X, \mathfrak{S}) and (Y, \mathfrak{S}') topological spaces, their expansive maps between them happen to be the open ones.

We do have now the following result,

THEOREM 3.5. *Let (X, A) and (X', A') be approach spaces and $f : X \rightarrow X'$ a bijective map. Then the following are equivalent:*

1. $f : (X, A) \rightarrow (X', A')$ is a contraction
2. $g(= f^{-1}) : (X', A') \rightarrow (X, A)$ is an expansion.

Proof. Let $x \in X$, $\varphi' \in A'(f(x))$. Then $x = g(f(x))$ and for all $z \in X$,

$$\begin{aligned} g(\varphi')(z) &= \inf_{g(y)=z} \varphi'(y) \\ &= \varphi'(f(z)) \\ &= (\varphi' \circ f)(z) \end{aligned}$$

and the claim $(\varphi' \circ f) \in A(x)$ is equivalent to $g(\varphi') \in A(g(f(x)))$. \square

Recalling now Theorem 3.15[3], the fact that the continuous image of a compact (countably compact, sequentially compact) topological space is compact (countably compact, sequentially compact), is trivially derived.

4. Case $p\mathbf{MET}^\infty$

We have seen that the properties for which you need countability are the same in **AP** so the main point is extended pseudometric approach spaces here and products which means completely regular spaces.

The classical results which simplify the study of compactness in pseudometric spaces, which proves that all three of the main kinds of compactness are identical, suggest a further study of the category $p\mathbf{MET}^\infty$.

Clearly for the $p\mathbf{MET}^\infty$ space X , the measures of countable and sequential compactness coincide and both will be smaller than the measure of compactness and we will be able to refine more their relation.

In order to achieve this, we will be using a mechanism to measure the deviation an approach space may have from being Lindelöf and from being precompact, that is, the canonical measures of Lindelöf and precompactness.

1. In [12] a *measure of precompactness* was defined in the uniform setting of approach theory. We rephrase this definition for spaces in $p\mathbf{MET}^\infty$ which are defined via a metric d ,

$$\mu_{pc}(X) = \inf_{Y \in 2^X} \sup_{z \in X} \inf_{y \in Y} d(y, z).$$

It was already shown in [12] but it is easily deduced from the above formula that, for $\epsilon < \mu_{pc}(X)$, no finite set Y can be found such that

$$X \subset \{d < \epsilon\}(Y)$$

and, thus, for the measure of compactness of the underlying approach space it holds

$$\begin{aligned} \mu_c(X) &= \sup_{\phi \in \prod_{z \in X} \Phi(z)} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{y \in Y} \phi(y)(z) \\ &= \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{y \in Y} d(y, z) \\ &= \mu_{pc}(X) \end{aligned}$$

2. [2] If X is an approach space then the measure of Lindelöf of X is defined as

$$L(X) = \sup_{\phi \in \prod_{z \in X} \Phi(z)} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{y \in Y} \phi(y)(z)$$

and for the $p\mathbf{MET}^\infty$ space (X, d) we have

$$L(X) = \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{y \in Y} d(y, z)$$

and, equivalently [2],

$$L(X) = \sup_{F \in \mathbb{F}_w(X)} \inf_{x \in X} \alpha F(x) = \sup_{F \in \mathbb{F}_w(X)} \inf_{x \in X} \sup_{F \in F} \inf_{y \in F} d(x, y)$$

where \mathbb{F}_w stands for the set of filters with the countable intersection property.

Analogously to the process done in Theorem 3.1, we obtain for the ∞ -pseudometric space X ,

$$(4.1) \quad \mu_{cc}(X) = \sup_{\lambda \in \Lambda} \inf_{K \in 2^{(\Gamma_\lambda)}} \sup_{z \in X} \inf_{k \in K} \inf_{x \in \lambda(k)} d(x, z)$$

where

$$\Lambda := \{\lambda : \Gamma_\lambda \subset \mathbb{N} \rightarrow 2^X \mid \forall x, \exists n_x \in \Gamma_\lambda : x \in \lambda(n_x)\}$$

and, further we have the following result.

THEOREM 4.1. *For an extended pseudometric approach space X , we have,*

$$\mu_{sc}(X) = \mu_{cc}(X) = L(X) = \mu_c(X).$$

Proof. Since, clearly, $\mathbb{F}_e(X) \subset \mathbb{F}_w(X)$ we can begin with:

$$\begin{aligned} \sup_{F \in \mathbb{F}_e(X)} \inf_{x \in X} \sup_{F \in F} \inf_{y \in F} d(x, y) &= \mu_{cc}(X) \\ &\leq \sup_{F \in \mathbb{F}_w(X)} \inf_{x \in X} \sup_{F \in F} \inf_{y \in F} d(x, y) = L(X) \\ &\leq \sup_{F \in \mathbb{F}(X)} \inf_{x \in X} \sup_{F \in F} \inf_{y \in F} d(x, y) = \mu_c(X) \end{aligned}$$

Secondly, as a consequence of (4.1), we have

$$L(X) \leq \mu_{cc}(X).$$

Thirdly, suppose that, for some $r \in \mathbb{R}_+$, there exists $F \in \mathbb{F}(X)$ such that

$$\inf_{x \in X} \sup_{F \in F} \inf_{y \in F} d(x, y) > r.$$

Then, for all $x \in X$, there exists $F_x \in F$:

$$d(x, y) > r, \quad \forall y \in F_x.$$

In [2] it was shown that, for a $pq\mathbf{MET}^\infty$ space X , we have,

$$L(X) = \sup_{x \in X} \inf_{y \in A} d(y, x), \quad \text{for a certain } A \in 2^{((X))}.$$

Then, in particular, for each $y \in A$ there exists $F_y \in F$ with $d(y, x) > r$, for all x in F_y and,

$$L(X) \geq \inf_{y \in A} \sup_{x \in X} d(y, x) > r$$

which proves that $\mu_c(X) \leq L(X)$.

Finally, the theorem is proved as we recall that, for a first countable approach space, the measures of countable and sequential compactness coincide. \square

References

- [1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories*, Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs, and Tracts, New York, ISBN 0-471-60922-6, MR1051419 (91h:18001) 1990

- [2] R. Baekeland and R. Lowen, *Measures of Lindelöf and separability in approach spaces*, Internat. J. Math. And Math. Sci. vol. **173** (1994), 597-606.
- [3] R. Baekeland and R. Lowen, *Measures of compactness in approach spaces*, Comm. Math. Univ. Carolinae, vol. **36** (1995), 327-345.
- [4] G. Choquet, *Convergences*, Ann. Univ. Grenoble, Sect. Math. Phys. vol. **23** (1948), 57-112.
- [5] D. Dikrajan and W. Tholen, *Cathegorical Structure of Closure Operators*, Mathematics and its Applications, Kluwer Academic Publishers MR1368854 (97i:18004), 1995.
- [6] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107-115.
- [7] S. P. Franklin, *Spaces in which sequences suffice II*, Fund. Math. **61** (1967), 51-56.
- [8] R. Lowen, *Kuratowski's measure of non-compactness revisited*, Quart. J. Math. Oxford **2** (1988), 235-254.
- [9] R. Lowen, *Approach Spaces. A common supercategory of **TOP** and **MET***, Math. Nachr. **141** (1995), 183-226.
- [10] R. Lowen, *Approach Spaces. The missing link in the Topology-Uniformity-Metric triad*, Oxford Science Publications, 1995.
- [11] R. Lowen and C. Verbeek, *Local compactness in approach spaces I*, Internat. J. Math. & Math. Sci. **21** (1998), no. 3, 429-438.
- [12] R. Lowen and B. Windels, *On the quantification of uniform properties*, Comment. Math. Univ. Carolinae **384** (1997), 749-759.
- [13] R. Lowen and B. Windels, ***A-Unif**: a common supercategory of **UNIF** and **MET***, Internat. J. Math. Sci. **21** (1998), 1-18.
- [14] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, MR0278261 (43:3992) 1970.
- [15] G. Preuss, *Theory of Topological Structures*, Mathematics and its Applications, Kluwer Academic Publishers MR0937052 (89m:54014) 1998.

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